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Asymptotically self-similar solutions to curvature flow equations with prescribed contact angle

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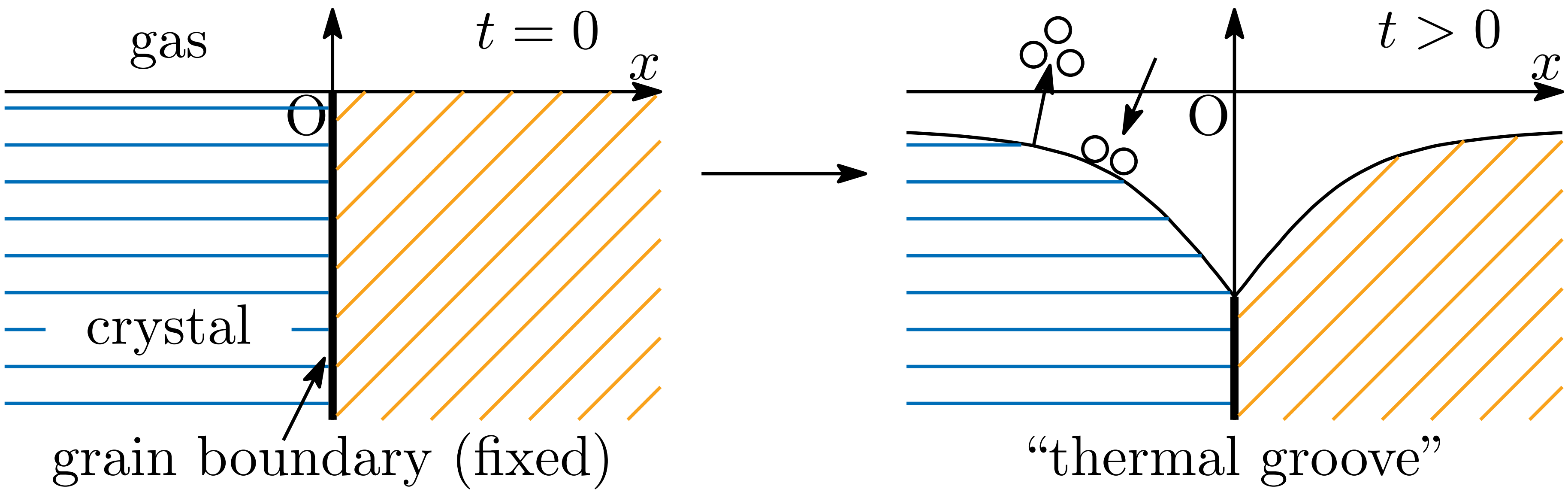
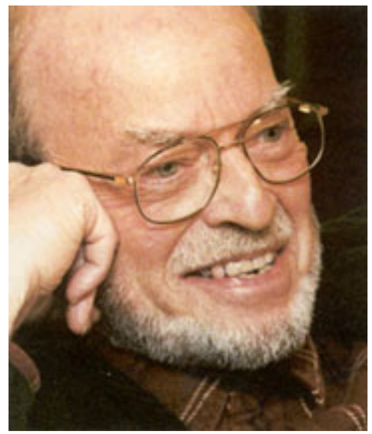
1 Introduction

Evaporation-condensation model

[Mullins ’57] **William W. Mullins** (1927–2001), Materials Scientist.





Mullins

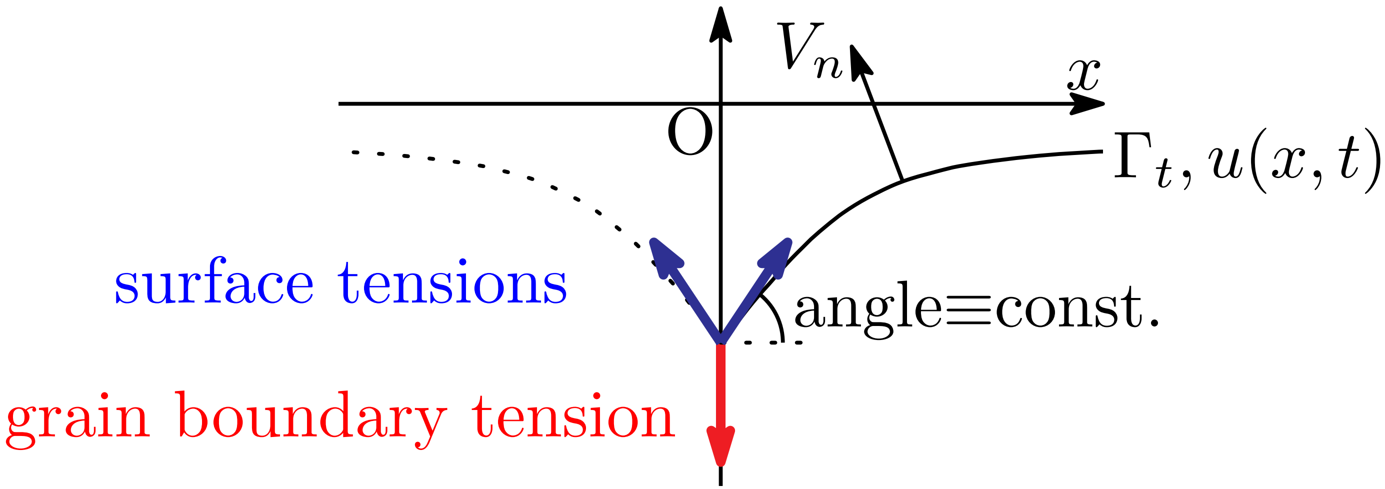
Surface diffusion model is also proposed in [Mullins ’57].
(: 4th order eq.)

$*$ Mg & high air pressure $\rightsquigarrow$ evaporation-condensation.
Au & low air pressure $\rightsquigarrow$ surface diffusion.

Equation and its derivation

$\Gamma_t = \{(x, u(x,t)) \in \mathbf{R}^2 \mid x \geqq 0, t \geqq 0\}$: surface (curve).

V_n: upward normal velocity. k: upward curvature.



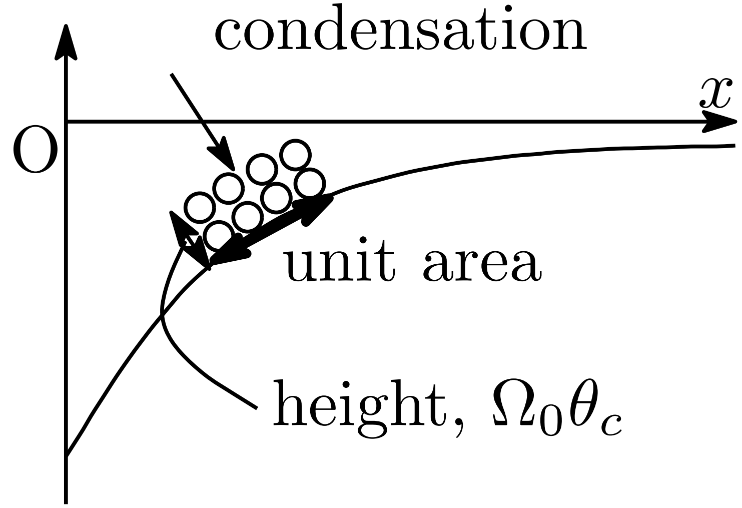
Generalized curvature flow equation: $V_n = 1 - e^{-k}$ on Γ_t, i.e.,

$$\boxed{\frac{u_t}{\sqrt{1+u_x^2}} = 1 - e^{-k}} \quad \text{with} \quad k = \frac{u_{xx}}{\sqrt{1+u_x^2}^3}.$$

Boundary condition: $\boxed{u_x(0,t) \equiv \beta} > 0$ by equilibrium of tensions.

Derivation.

- Upward normal velocity V_n.



$$\begin{aligned} V_n &= (\text{condensation}) - (\text{evaporation}) \\ &= \Omega_0\theta_c - \Omega_0\theta_e \\ &= \Omega_0 \cdot C_1(p_0 - p). \quad (C_1 > 0) \qquad (*1) \end{aligned}$$

* Here Ω_0: molecular volume,

θ_c (θ_e): number of impinging (emitted) atoms per unit time and unit area,

p_0 (p): vapor pressure in the atmosphere (in equilibrium with the surface).

- **Gibbs-Thompson formula**: $\boxed{\log \frac{p}{p_0} = -C_2 k}$ $(C_2 > 0)$. $\qquad (*2)$

(k: upward curvature)

$$(*1)\ \&\ (*2) \implies V_n = \Omega_0 C_1 p_0 \left(1 - e^{-C_2 k}\right).$$

Approximations by Mullins

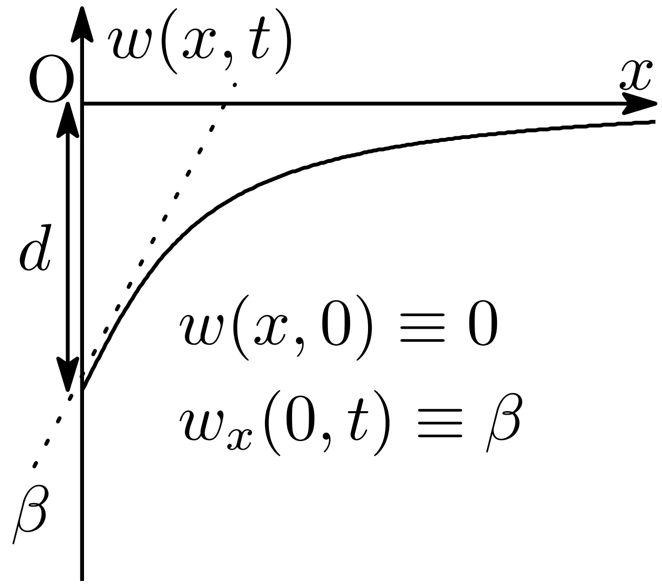
$u(x,0) \equiv 0,\ u_x(0,t) \equiv \beta \ll 1.$

$$\boxed{\frac{u_t}{\sqrt{1+u_x^2}} = 1 - e^{-k}} \overset{1-e^{-k}\approx k}{\dashrightarrow} \boxed{v_t = \frac{v_{xx}}{1+v_x^2}} \overset{v_x\approx 0}{\dashrightarrow} \boxed{w_t = w_{xx}}$$

generalized curvature flow eq.   curvature flow eq. for graph   heat eq.

Solving the heat equation, Mullins concludes the groove profile is

$$\boxed{w(x,t) = -2\beta\sqrt{t}\cdot \mathrm{ierfc}\left(\frac{x}{2\sqrt{t}}\right)}.$$



In particular, the depth at the origin is

$$d := -w(0,t) = 2\beta\sqrt{\frac{t}{\pi}} \approx 1.13\beta\sqrt{t}.$$

$*$ Here $\mathrm{ierfc}(x)$ is the **integral error function**:

$$\mathrm{ierfc}(x) = \int_x^\infty \mathrm{erfc}(z)dz, \quad \mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-z^2}dz.$$

Goal

- **Justification of Mullins' two approximations.**

$$\boxed{\frac{u_t}{\sqrt{1+u_x^2}} = 1 - e^{-k}} \overset{1-e^{-k}\approx k}{\dashrightarrow} \boxed{v_t = \frac{v_{xx}}{1+v_x^2}} \overset{v_x\approx 0}{\dashrightarrow} \boxed{w_t = w_{xx}}$$

Important remark. v & w are self-similar, i.e.,

$$v(x,t) = \sqrt{t}V\left(\frac{x}{\sqrt{t}}\right), \quad w(x,t) = \sqrt{t}W\left(\frac{x}{\sqrt{t}}\right).$$

(V, W: profile functions.)

Results. (1) $u \approx v$? $\star$ u is asymptotically self-similar, i.e.,

$$\frac{1}{\sqrt{t}}u(\sqrt{t}x,t) \xrightarrow{t\to\infty} V(x).$$

(2) $v \approx w$? $\star$ $V(0) = W(0) + O(\beta^{1+2})$ as $\beta \to 0$. (Two depths)

Related work

$$\frac{u_t}{\sqrt{1+u_x^2}} = 1 - e^{-k} \quad (1) \quad \overset{1-e^{-k}\approx k}{\dashrightarrow} \quad v_t = \frac{v_{xx}}{1+v_x^2} \quad (2) \quad \overset{v_x \approx 0}{\dashrightarrow} \quad w_t = w_{xx} \quad (3)$$

• [Broadbridge '89] Exact solvability of (2) on $\{x \geqq 0\} \times \{t \geqq 0\}$ with $v(x,0) \equiv 0$, $v_x(0,t) \equiv \beta$.

• [Ogasawara '03 (J. Phys. Soc. Jpn.)] Generalized model under a temperature gradient. Existence of stationary solutions.

• [Alber-Zhu '07] Solvability of (2) on $\{a \leqq x \leqq b\} \times (0,\infty)$ and asymptotics. Weak, strong and classical solutions.

• [Nara-Taniguchi '07] Let v and w be, resp., solutions to (2) and (3) in $\mathbf{R} \times (0,\infty)$ with the same initial data. Then

$$\sup_{\mathbf{R}} |v(\cdot,t) - w(\cdot,t)| = O(1/\sqrt{t}) \text{ as } t \to \infty.$$

∗ A similar convergence result does not hold in our case.

$$\sup_{[0,\infty)} |v(\cdot,t) - w(\cdot,t)| = \sqrt{t} \sup_{[0,\infty)} |V(\cdot) - W(\cdot)| \overset{t\to\infty}{\longrightarrow} \infty$$

for $v(x,t) = \sqrt{t}V(x/\sqrt{t})$ and $w(x,t) = \sqrt{t}W(x/\sqrt{t})$ such that $v \not\equiv w$.

2 Neumann boundary problems

Let $F : \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$ be continuous & degenerate elliptic.

$$
\text{(NP)} \quad \begin{cases} u_t(x,t) = F(\nabla_x u(x,t), \nabla_x^2 u(x,t)) & \text{in } \{x_1 > 0\} \times (0,\infty), \\ u(x,0) = u_0(x) \in BUC & \text{on } \{x_1 \geqq 0\}, \\ u_{x_1}(x,t) = \beta > 0 & \text{on } \{x_1 = 0\} \times (0,\infty). \end{cases}
$$

Theorem. (NP)=(NP; F, u_0) admits a unique viscosity solution which is bounded on $\{x_1 \geqq 0\} \times [0, \forall T)$.

∗ The boundary condition is interpreted as the viscosity sense.

cf. (Neumann problems and viscosity sol.)

- [Lions '85] pioneer.
- [Barles '99], [Ishii-Sato '04] general singular 2nd order eq.

} bounded domain

- [Sato '96] half space, capillary boundary condition: $u_{x_1} - k|\nabla u| = 0$.

3 Asymptotic behavior

$$u_t(x,t) = F(\nabla_x u(x,t), \nabla_x^2 u(x,t))$$

Mullins' case. $(n=1)$

$$G_M(p,X) = \sqrt{1+p^2}\left(1 - e^{-X/\sqrt{1+p^2}^3}\right), \qquad F_M(p,X) = \frac{X}{1+p^2}.$$

Definition (Homogeneity). $F, G : \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$.

- F: homogeneous (hom.)

  $\overset{\text{def.}}{\Longleftrightarrow} \lambda F(p, X/\lambda) = F(p,X), \ \forall \lambda > 0.$

- G: asymptotically homogeneous (a-hom.)

  $\overset{\text{def.}}{\Longleftrightarrow} \exists \tilde{F}$: hom., $\lambda G(p, X/\lambda) \overset{\lambda\to\infty}{\longrightarrow} \tilde{F}(p,X)$ loc. unif. in $\mathbf{R}^n \times \mathbf{S}^n$.

$*$ G_M is a-hom. with the limit F_M.

$\star$ Generalized Mullins' 1st approx.

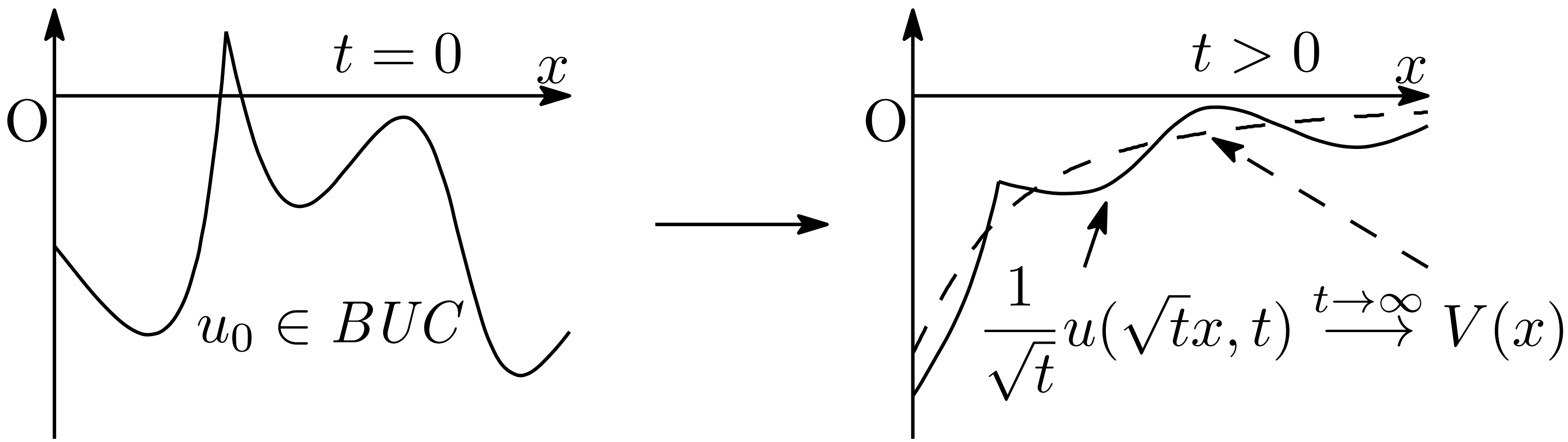
$$\boxed{G \approx F}$$

Theorem (Asymptotic convergence).
Assume G is a-hom. with the limit F.
Let u solve (NP; G, u_0), v solve (NP; $F, 0$) (self-similar). Then

$$u_{(\lambda)}(x,t) := \frac{1}{\lambda} u(\lambda x, \lambda^2 t) \xrightarrow{\lambda\to\infty} v(x,t)$$

locally uniformly on $\{x_1 \geqq 0\} \times [0, \infty)$.

Remark. The limit is common to all initial data.



(V is the profile function of v.)

Proof. (u solves (NP; G,u_0), v solves (NP; F,0).)

1.

$$\star\ u_{(\lambda)} \text{ is a solution of (NP; } \underbrace{\lambda G(p, X/\lambda)}_{\to F}, \underbrace{u_0(\lambda x)/\lambda}_{\to 0}).$$

Thus $u_{(\lambda)} \to v$ as $\lambda \to \infty$ if the limit of $u_{(\lambda)}$ exists.

We employ the viscosity solution theory to show $u_{(\lambda)} \to v$.

Relaxed limits:

$$\begin{cases} \overline{u} := \limsup^*_{\lambda\to\infty} u_{(\lambda)} & \text{is a subsol. of (NP; } F, 0), \\ \underline{u} := \liminf_{*\lambda\to\infty} u_{(\lambda)} & \text{is a supersol. of (NP; } F, 0). \end{cases}$$

These limits exist if $\{u_{(\lambda)}\}_\lambda$ is locally uniformly bounded. Then

$$\boxed{\overline{u} \geq \underline{u}} \text{ by definition,} \quad \boxed{\overline{u} \leq \underline{u}} \text{ by comparison principle.}$$

Thus $\overline{u} = \underline{u} = v$, which also implies the locally uniform convergence.

2. We construct barriers $\Phi^\pm$ such that

$$\Phi^- \leqq u \leqq \Phi^+ \quad \& \quad \{\Phi^\pm_{(\lambda)}\}_\lambda \text{ are locally bounded.}$$

$\boxed{\Phi^-}$ We define $\Phi^-(x,t) := -C + w(x_1,t) - g(t)$, where $C \gg 1$, w is a solution of the heat equation and

$$g'(t) = \sup_{|\theta|,|\sigma|\leqq 1} \left| G\left(\theta\beta e_1, \frac{\sigma}{\sqrt{t}} I_{1,1}\right)\right| \quad (t>1).$$

- (A) Φ^- is a subsolution. $(\Longrightarrow \Phi^- \leqq u.)$

Since $w_t \leqq 0$, $0 \leqq w_{x_1} \leqq \beta$ and $-1/\sqrt{t} \leqq w_{x_1x_1} \leqq 0$, we see

$$g'(t) \geqq -G((w_{x_1})e_1, (w_{x_1x_1})I_{1,1}) + w_t.$$

- (B) $g(t) = O(\sqrt{t})$ as $t \to \infty$. $(\Longrightarrow \{\Phi^-_{(\lambda)}\}_\lambda$ is locally bounded.)

If G is hom., $g'(t) = (\text{const.})/\sqrt{t}$. Thus $g(t) = O(\sqrt{t})$. General cases:

$$\sqrt{t}g'(t) \leqq \underbrace{\sup\left|\sqrt{t}G\left(\cdot\,, \frac{\cdot}{\sqrt{t}}\right) - F\right|}_{\to 0\ (t\to\infty)} + \underbrace{\sup|F|}_{=\text{const.}} \leqq \text{const.} \quad \square$$

Remark. If G is hom., then

$$u_{(\lambda)} \xrightarrow{\lambda\to\infty} v \quad \text{~~locally~~ uniformly on } \overline{\Omega}\times[0,\infty).$$

(u solves (NP; $G \equiv F, u_0$), v solves (NP; $F, 0$).)

($\because$) **Contraction property**:

$$u_{01}, u_{02} \in BUC(\overline{\Omega}), \text{ two initial data,}$$
$$u_1\text{: sol. of (NP; } F, u_{01}) \ \&\ u_2\text{: sol. of (NP; } F, u_{02}).$$
$$\Longrightarrow \|u_1 - u_2\|_{L^\infty(\overline{\Omega}\times[0,\infty))} \leqq \|u_{01} - u_{02}\|_{L^\infty(\overline{\Omega})}.$$

Letting $(u_1, u_{01}) = (u_{(\lambda)}, u_0(\lambda x)/\lambda)$ and $(u_2, u_{02}) = (v, 0)$, we see

$$\|u_{(\lambda)} - v\|_{L^\infty} \leqq \|u_0(\lambda x)/\lambda - 0\|_{L^\infty} = \frac{1}{\lambda}\|u_0\|_{L^\infty} \xrightarrow{\lambda\to\infty} 0.$$

Asymptotics of solutions to curvature flow type eq.

Neumann type conditions.

- [Huisken '89] Convergence to a constant, zero Neumann.
- [Altschuler-Wu '93] 1-dim, quasilinear, non-zero Neumann.
  [Altschuler-Wu '94] 2-dim, curvature flow, non-zero Neumann.
- [Ishimura '95] Opening angle: $u_x(-\infty) = -K_2$, $u_x(\infty) = K_1$.
- [Deckelnick-Elliott-Richardson '97] 1-dim half-space, Driving force.
- [Kohsaka '01], [Chang-Guo-Kohsaka '03] Free boundary, quasilinear.

Others.

- [Ecker-Huisken '89] Entire graphs.
- [Ishii-Pires-Souganidis '99] Level set.
- [Chen-Guo '07], [Schnürer-Schulze '07] Triple junction.

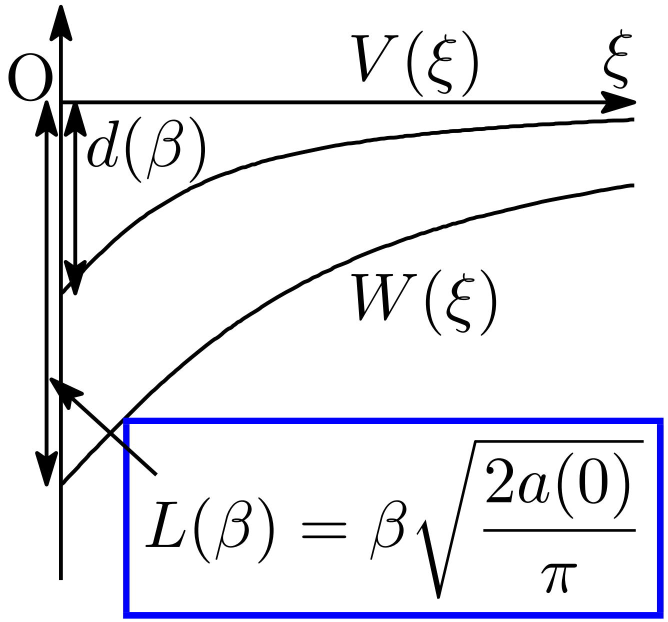
4 Depth of the groove

Let $n = 1$. The profile function V satisfies

$$\text{(ODE)} \quad \begin{cases} V(\xi) - \xi V'(\xi) = \underline{\underline{a(V'(\xi))}} V''(\xi) \quad \text{in } (0, \infty), \\ V'(0) = \beta > 0, \\ \lim_{\xi \to \infty} V(\xi) = 0, \end{cases}$$

where $a(p) := -2F(p, -1)$. (F is homogeneous.) We also consider

$$\text{(LODE)} \quad \begin{cases} W(\xi) - \xi W'(\xi) = \underline{\underline{a(0)}} W''(\xi) \quad \text{in } (0, \infty), \\ W'(0) = \beta > 0, \\ \lim_{\xi \to \infty} W(\xi) = 0. \end{cases}$$



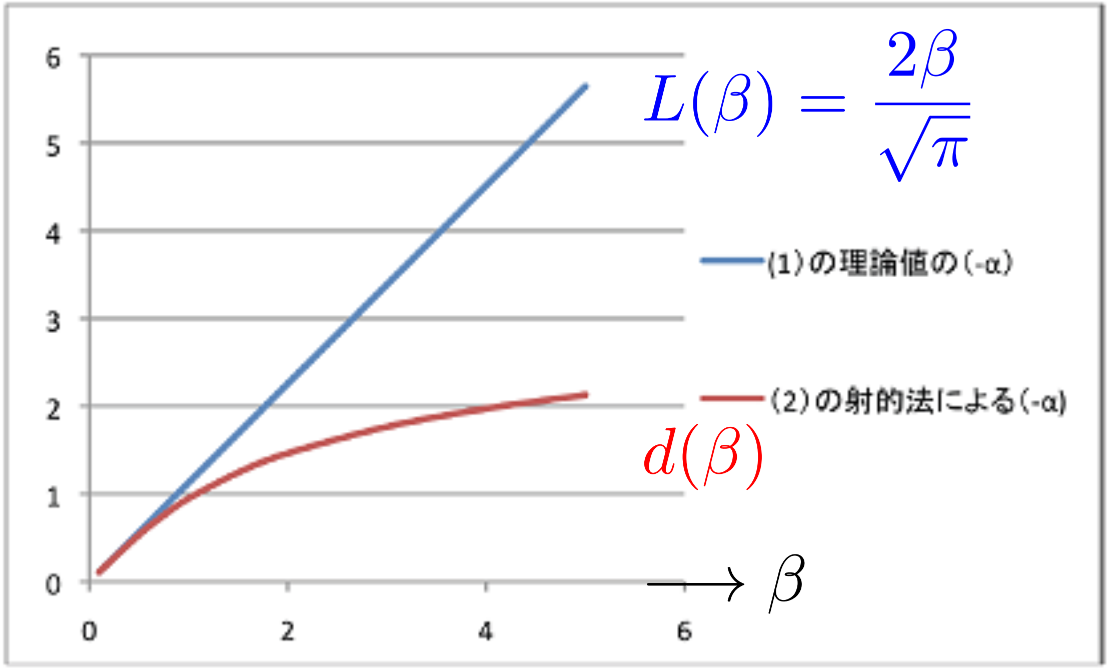
Set $d(\beta) := -V(0)$ and $L(\beta) := -W(0)$.

⋆ Generalized Mullins' 2nd approx.

$$\boxed{a(V'(\xi)) \approx a(0)}$$

Numerical result. Mullins' case, i.e., $a(p) := \dfrac{2}{1+p^2}$.

$$\boxed{v_t = \frac{v_{xx}}{1+v_x^2}} \overset{\text{depth}}{\longrightarrow} d(\beta) = -v(0,1), \ \boxed{w_t = w_{xx}} \overset{\text{depth}}{\longrightarrow} L(\beta) = -w(0,1).$$



[Yamazaki, '11, graduation research]

▷ Is $L(\beta)$ a linear approximation of $d(\beta)$ at $\beta = 0$?

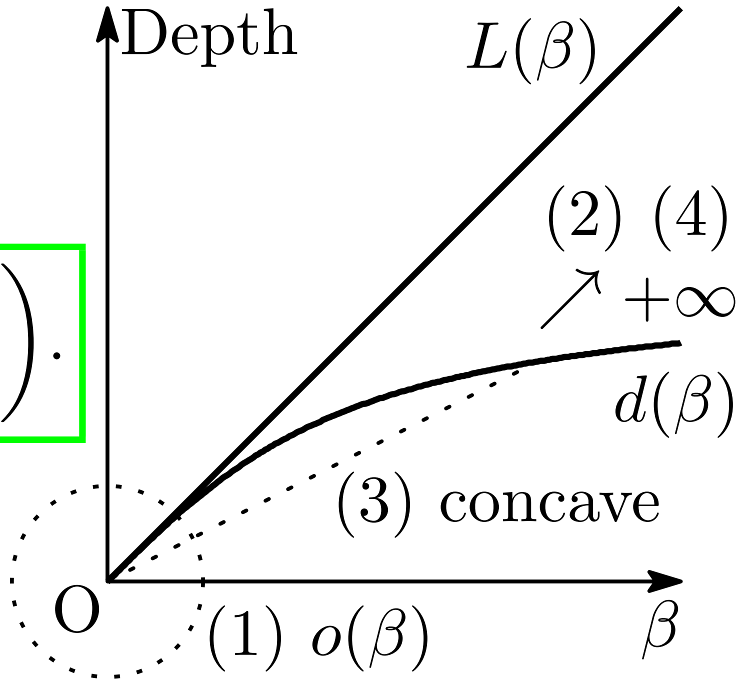
▷ $d(\beta) \leq L(\beta)$? Is $d(\beta)$ increasing, concave?

▷ Does $d(\beta)$ go to $+\infty$?

Theorem (Depth of the groove).

Assume $0 \leqq a(p) \leqq a(0)$ $(\forall p \geqq 0)$. Then

(1) $$0 \leqq \frac{L(\beta) - d(\beta)}{\beta} \leqq {}^{\exists}C\left(a(0) - \min_{[0,\beta]} a\right).$$



We also have

(2) d is nondecreasing in $(0, \infty)$.

(3) $\lambda d(\beta) \leqq d(\lambda\beta)$ $(\forall \lambda \in [0,1])$ if a is nonincreasing.

(4) $\lim_{\beta \to \infty} d(\beta) = +\infty$ if $a(p) \geqq \frac{c}{1+p^2}$ $(\forall p \gg 1)$.

Mullins' case. $a(p) = 2/(1+p^2)$.

$$a(0) - \min_{[0,\beta]} a = 2 - \frac{2}{1+\beta^2} = \frac{2\beta^2}{1+\beta^2} = O(\beta^2) \quad \text{as } \beta \to 0.$$

Proof. **Comparison principle.**

(1) $\{L(\beta)-d(\beta)\}/\beta \leqq C\left(a(0)-\min_{[0,\beta]} a\right)$. We claim

$$\boxed{d(\beta) \geqq \beta\sqrt{\frac{2\min_{[0,\beta]} a}{\pi}}}. \qquad (\#)$$

Take $\beta_0 > 0$ such that $a(\beta_0) = \min_{[0,\beta]} a \ (>0)$. Let U solve

$$\text{(LODE)} \quad U - \xi U' = a(\beta_0)U'' \quad \& \quad \text{B.C.}$$

Then, since $0 \leqq U' \leqq \beta$ ($\Longrightarrow a(U') \geqq a(\beta_0)$) and $U'' \leqq 0$, we see

$$U - \xi U' = a(\beta_0)U'' \geqq a(U')U'',$$

which implies U is a supersol. of (ODE). Thus $V(\xi) \leqq U(\xi)$ by the comparison principle, and putting $\xi = 0$ yields (#). By (#)

$$\frac{L(\beta)-d(\beta)}{\beta} \leqq \sqrt{\frac{2a(0)}{\pi}} - \sqrt{\frac{2a(\beta_0)}{\pi}} = \sqrt{\frac{2}{\pi}} \times \frac{a(0)-a(\beta_0)}{\sqrt{a(0)}+\sqrt{a(\beta_0)}}. \qquad \square$$

5 Surface diffusion equation

$$\begin{cases} u_t = -\partial_x \left[\dfrac{1}{\sqrt{1+u_x^2}} \partial_x \left(\dfrac{u_{xx}}{\sqrt{1+u_x^2}^3} \right) \right] & \text{in } \{x>0\} \times (0,\infty), \quad (1) \\ u(x,0) \equiv 0 & \text{on } \{x \geqq 0\}, \\ u_x(0,t) = \beta > 0 & \text{in } (0,\infty), \\ \partial_x \left(\dfrac{u_{xx}}{\sqrt{1+u_x^2}^3} \right) \Bigg|_{x=0} = 0 & \text{in } (0,\infty). \quad (2) \end{cases}$$

$(1) \iff V_n = -k_{ss}$, (2) **No flux condition**.

$\star$ Comparison principle (maximum principle) does not hold.

$*$ ~~Viscosity solution theory~~ (n-th order, $n \geqq 3$)

Linearization. $u_x \approx 0$. $(1) \rightsquigarrow y_t = -y_{xxxx}$, $(2) \rightsquigarrow y_{xxx}(0,t) = 0$.

[Martin '09] Exact solution to the linearized problem.